

# SIXTH TERM EXAMINATION PAPERS

administered by the Oxford and Cambridge Schools Examination Board  
on behalf of the Cambridge Colleges

9465

## MATHEMATICS I

Friday 1 July 1994, afternoon

3 hours

*Additional materials:*

*script paper; graph paper; MF(STEP)1.*

*To be brought by candidate: electronic calculator;  
standard geometrical instruments.*

*All questions carry equal weight.*

*You are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.*

*You may attempt as many questions as you wish with no restriction of choice but marks will be assessed on the six questions best answered.*

*You are provided with Mathematical Formulae and Tables MF(STEP)1.*

*The use of electronic calculators is permitted.*

### Section A: Pure Mathematics

**1** My house has an attic consisting of a horizontal rectangular base of length  $2q$  and breadth  $2p$  (where  $p < q$ ) and four plane roof sections each at angle  $\theta$  to the horizontal. Show that the length of the roof ridge is independent of  $\theta$  and find the volume of the attic and the surface area of the roof.

**2** Given that  $a$  is constant, differentiate the following expressions with respect to  $x$ :

- (i)  $x^a$ ;
- (ii)  $a^x$ ;
- (iii)  $x^x$ ;
- (iv)  $x^{(x^x)}$ ;
- (v)  $(x^x)^x$ .

**3** By considering the coefficient of  $x^n$  in the identity  $(1-x)^n(1+x)^n = (1-x^2)^n$ , or otherwise, simplify

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \binom{n}{3}^2 + \cdots + (-1)^n \binom{n}{n}^2$$

in the cases (i) when  $n$  is even, (ii) when  $n$  is odd.

**4** Show that

(i)  $\frac{1 - \cos \alpha}{\sin \alpha} = \tan \frac{1}{2}\alpha,$

(ii) if  $|k| < 1$  then  $\int \frac{1}{1 - 2kx + x^2} dx = \frac{1}{\sqrt{1 - k^2}} \tan^{-1} \frac{x - k}{\sqrt{1 - k^2}} + C,$  where  $C$  is a constant of integration.

Hence, or otherwise, show that if  $0 < \alpha < \pi$  then

$$\int_0^1 \frac{\sin \alpha}{1 - 2x \cos \alpha + x^2} dx = \frac{1}{2}(\pi - \alpha).$$

**5** A parabola has the equation  $y = x^2$ . The points  $P$  and  $Q$  with coordinates  $(p, p^2)$  and  $(q, q^2)$  respectively move on the parabola in such a way that  $\angle POQ$  is always a right angle.

- (i) Find and sketch the locus of the midpoint  $R$  of the chord  $PQ$ .
- (ii) Find and sketch the locus of the point  $T$  where the tangents to the parabola at  $P$  and  $Q$  intersect.

**6** The function  $f$  is defined, for any complex number  $z$ , by

$$f(z) = \frac{iz - 1}{iz + 1}.$$

Suppose throughout that  $x$  is a real number.

(i) Show that

$$\operatorname{Re} f(x) = \frac{x^2 - 1}{x^2 + 1} \quad \text{and} \quad \operatorname{Im} f(x) = \frac{2x}{x^2 + 1}.$$

(ii) Show that  $f(x)f(x)^* = 1$ , where  $f(x)^*$  is the complex conjugate of  $f(x)$ .

(iii) Find expressions for  $\operatorname{Re} f(f(x))$  and  $\operatorname{Im} f(f(x))$ .

(iv) Find  $f(f(f(x)))$ .

**7** From the facts

$$\begin{aligned} 1 &= 0 + 1 \\ 2 + 3 + 4 &= 1 + 8 \\ 5 + 6 + 7 + 8 + 9 &= 8 + 27 \\ 10 + 11 + 12 + 13 + 14 + 15 + 16 &= 27 + 64 \end{aligned}$$

guess a general law. Prove it.

Hence, or otherwise, prove that

$$1^3 + 2^3 + 3^3 + \cdots + N^3 = \frac{1}{4}N^2(N + 1)^2$$

for every positive integer  $N$ .

[You may assume that  $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$ .]

**8** By means of the change of variable  $\theta = \frac{1}{4}\pi - \phi$ , or otherwise, show that

$$\int_0^{\pi/4} \ln(1 + \tan \theta) d\theta = \frac{\pi}{8} \ln 2.$$

Evaluate  $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$  and  $\int_0^{\pi/2} \ln\left(\frac{1+\sin x}{1+\cos x}\right) dx$ .

### Section B: Mechanics

**9** A cannon-ball is fired from a cannon at an initial speed  $u$ . After time  $t$  it has reached height  $h$  and is at a distance  $\sqrt{x^2 + h^2}$  from the cannon. Ignoring air resistance, show that

$$\frac{1}{4}g^2t^4 - (u^2 - gh)t^2 + h^2 + x^2 = 0.$$

Hence show that if  $u^2 > 2gh$  then the horizontal range for a given height  $h$  and initial speed  $u$  is less than or equal to

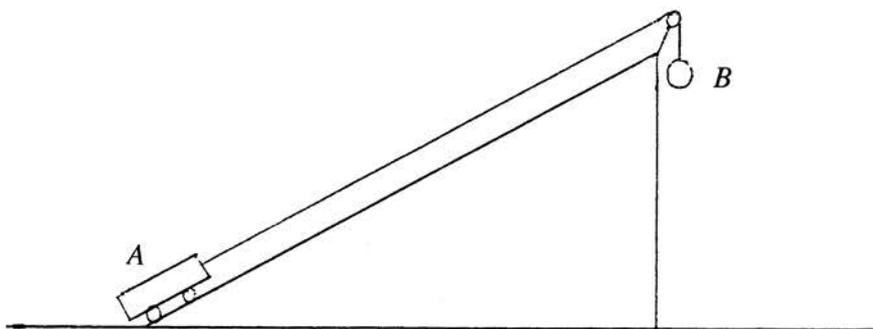
$$\frac{u\sqrt{u^2 - 2gh}}{g}.$$

Show that there is always an angle of firing for which this value is attained.

**10** One end  $A$  of a light elastic string of natural length  $l$  and modulus of elasticity  $\lambda$  is fixed and a particle of mass  $m$  is attached to the other end  $B$ . The particle moves in a horizontal circle with centre on the vertical through  $A$  with angular velocity  $\omega$ . If  $\theta$  is the angle  $AB$  makes with the downward vertical, find an expression for  $\cos \theta$  in terms of  $m$ ,  $g$ ,  $l$ ,  $\lambda$  and  $\omega$ .

Show that the motion described is possible only if

$$\frac{g\lambda}{l(\lambda + mg)} < \omega^2 < \frac{\lambda}{ml}.$$



**11** The diagram shows a small railway wagon  $A$  of mass  $m$  standing at the bottom of a smooth railway track of length  $d$  inclined at an angle  $\theta$  to the horizontal. A light inextensible string, also of length  $d$ , is connected to the wagon and passes over a light frictionless pulley at the top of the incline. On the other end of the string is a ball  $B$  of mass  $M$  which hangs freely. The system is initially at rest and is then released.

(i) Find the condition which  $m$ ,  $M$  and  $\theta$  must satisfy to ensure that the ball will fall to the ground. Assuming that this condition is satisfied, show that the velocity  $v$  of the ball when it hits the ground satisfies

$$v^2 = \frac{2g(M - m \sin \theta)d \sin \theta}{M + m}.$$

(ii) Find the condition which  $m$ ,  $M$  and  $\theta$  must satisfy if the wagon is not to collide with the pulley at the top of the incline.

### Section C: Probability

**12** There are 28 colleges in Cambridge, of which two (New Hall and Newnham) are for women only; the others admit both men and women. Seven women, Anya, Betty, Celia, Doreen, Emily, Fariza and Georgina, are all applying to Cambridge. Each has picked three colleges at random to enter on her application form.

- (i) What is the probability that Anya's first choice college is single-sex?
- (ii) What is the probability that Betty has picked Newnham?
- (iii) What is the probability that Celia has picked at least one single-sex college?
- (iv) Doreen's first choice college is Newnham. What is the probability that one of her other two choices is New Hall?
- (v) Emily has picked Newnham. What is the probability that she has also picked New Hall?
- (vi) Fariza's first choice college is single-sex. What is the probability that she has also chosen the other single-sex college?
- (vii) One of Georgina's choices is a single-sex college. What is the probability that she has also picked the other single-sex college?

**13** I have a bag containing  $M$  tokens,  $m$  of which are red. I remove  $n$  tokens from the bag at random without replacement. Let  $X_i = 1$  if the  $i$ th token I remove is red; let  $X_i = 0$  if not. Let  $X$  be the total number of red tokens I remove.

- (i) Explain briefly why  $X = X_1 + X_2 + \dots + X_n$ .
- (ii) Find the expectation  $E(X_i)$ .
- (iii) Show that  $E(X) = \frac{mn}{M}$ .
- (iv) Find  $P(X = k)$  for  $k = 0, 1, 2, \dots, n$ .
- (v) Deduce that  $\sum_{k=1}^n k \binom{m}{k} \binom{M-m}{n-k} = m \binom{M-1}{n-1}$ .

**14** Each of my  $n$  students has to hand in an essay to me. Let  $T_i$  be the time at which the  $i$ th essay is handed in and suppose that  $T_1, T_2, \dots, T_n$  are independent, each with probability density function  $\lambda e^{-\lambda t}$  ( $t \geq 0$ ). Let  $T$  be the time I receive the first essay to be handed in and let  $U$  be the time I receive the last one.

- (i) Find the mean and variance of  $T_i$ .
- (ii) Show that  $P(U \leq u) = (1 - e^{-\lambda u})^n$  for  $u \geq 0$ , and hence find the probability density function of  $U$ .
- (iii) Obtain  $P(T > t)$ , and hence find the probability density function of  $T$ .
- (iv) Write down the mean and variance of  $T$ .